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Open string star as a continuous Moyal product

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Abstract: We establish that the open string star product in the zero momentum sector can be described as a continuous tensor product of mutually commuting two dimensional Moyal star products. Let the continuous variable $\kappa \in [0, \infty)$ parametrize the eigenvalues of the Neumann matrices; then the noncommutativity parameter is given by $\theta(\kappa) = 2\tanh(\pi\kappa/4)$. For each κ, the Moyal coordinates are a linear combination of even position modes, and the Fourier transform of a linear combination of odd position modes. The commuting coordinate at $\kappa = 0$ is identified as the momentum carried by half the string. We discuss the relation to Bars' work, and attempt to write the string field action as a noncommutative field theory.

Keywords: Bosonic Strings, String Field Theory, Non-Commutative Geometry.

Contents

1. Introduction and summary

The star product of open string functionals is a noncommutative associative product based on a simple prescription for the gluing of the underlying open strings [1]. In this prescription, the right-half of the first string must coincide with the left-half of the second string, and the resulting string is composed by the left-half of the first string together with the right-half of the second string. As with pointwise multiplication of functions, where position space delta functions define the kernel imposing the coincidence conditions, the open string star product uses delta functionals of half-strings to impose the requisite coincidence conditions. The language of delta functionals, however, is difficult to use in explicit computations and the methods of conformal field theory were used to develop Fock space descriptions of the star product as a three string vertex [2, 3]. While these concrete constructions have allowed much recent progress in open string field theory, at a foundational level important questions remain. The conditions that specify the space of open string fields for which the axioms of the algebra hold are not yet known. Moreover, the star algebra has not been given a structural definition in terms of well-understood algebras.

In a stimulating paper Bars [4] set out to describe the open string star product in terms of the Moyal product. This product is the unique associative deformation (up to isomorphism) of the pointwise multiplication of functions on $\mathbb{R}^{2n}$, and as such has played a central role in recent studies and constructions of noncommutative field theories (see, for example [5, 6, 7]). Bars proposed that each even position mode (except for the zero mode) together with a specific linear combination of the Fourier transforms of the odd position modes forms a Moyal pair. The various Moyal pairs are mutually commuting and they all have the same noncommutativity parameter $\theta \neq 0$. The precise treatment of the center-of-mass or string midpoint was left for future work.

In this paper, we investigate the description of the open string star product in terms of Moyal products using a different starting point. Recently, the Neumann matrices defining the oscillator form of the three-open-string vertex have been diagonalized and the spectrum and eigenvectors have been constructed explicitly [9]. In addition, orthonormality and completeness of the eigenvectors was proven in [10]. These results imply that there is a basis of oscillators where the exponential in the vertex takes diagonal form, and thus furnishes a representation of the product in terms of mutually commuting algebras.

We find that each commuting factor in this product is a Moyal product, and thus each algebra is a (one dimensional) Heisenberg algebra. Since the spectrum of the Neumann matrices is continuous, we get a continuous tensor product of Heisenberg algebras with a smoothly varying noncommutativity parameter. It should be emphasized that we work on the subspace of zero momentum functionals, that is, functionals that are independent of the center of mass coordinate of the string.

Our Moyal coordinates are constructed as follows. For each $\kappa \in (-\infty, \infty)$ the Neumann matrices M, M^{12}, and M^{21} have a common eigenvector $v(\kappa)$ with eigenvalues $\mu(\kappa), \mu^{12}(\kappa)$ and $\mu^{21}(\kappa)$ respectively. The signal of noncommutativity is the fact that $\mu^{12}(\kappa) \neq \mu^{21}(\kappa)$. Under the action of open string twist, the eigenvectors $v(\kappa)$ and $v(-\kappa)$ mix, and one can conveniently define twist odd and twist even eigenvectors for $\kappa \in [0, \infty)$. The unpaired eigenvector at $\kappa = 0$ is actually twist odd, and has featured in several studies of vacuum string field theory and sliver states [11]–[14]. It will have an important role here as well. The definite twist combinations are degenerate eigenvectors of M, and while they not eigenvectors of M^{12} and M^{21}, these matrices have simple action on them. We find that for each $\kappa > 0$, the twist even and the twist odd eigenvectors respectively define the Moyal coordinates $x(\kappa)$ and $y(\kappa)$, with noncommutativity parameter given as[1]

$$\theta(\kappa) = 2 \tanh\left(\frac{\pi \kappa}{4}\right), \qquad \kappa \geq 0\,. \tag{1.1}$$

Note that for $\kappa = 0$, where there is only one eigenvector, and thus only one coordinate, there is no scope for a Moyal product, and consistently the noncommutativity vanishes. The various coordinates for different values of κ commute, so that we have

$$[x(\kappa), y(\kappa')]_* = i\theta(\kappa)\delta(\kappa - \kappa')\,. \tag{1.2}$$

The description of the star product in terms of the above Moyal products is as follows. Given a string field $\Psi(X(\sigma))$, we view it as a function $\Psi(\{x_{2n}\}, \{x_{2n-1}\})$ of the even and odd modes in the expansion of $X(\sigma)$.
Then we Fourier transform Ψ on the odd modes x_{2n-1} into variables p_{2n-1} (that correspond to the eigenvalues of $\hat{p}_{2n-1}$) obtaining a functional $\widetilde{\Psi}(\{x_{2n}\}, \{p_{2n-1}\})$. This functional is

[1] With units inserted back there is an α' factor in the right hand side of (1.1). So, as usual, θ has dimensions of (length)^2.

now writen in terms of coordinates $(x(\kappa), y(\kappa))$ using the invertible relations

$$x(\kappa) = \sqrt{2}\sum_{n=1}^{\infty} v_{2n}(\kappa)\sqrt{2n}x_{2n}\,, \qquad y(\kappa) = -\sqrt{2}\sum_{n=1}^{\infty} \frac{v_{2n-1}(\kappa)}{\sqrt{2n-1}}p_{2n-1}\,, \tag{1.3}$$

that, as mentioned before, use the twist even and twist odd eigenvectors. The resulting field $\Psi^M(x(\kappa), y(\kappa))$ is star multiplied just using the Moyal product on the underlying coordinates.[2] As we will see, this continuous Moyal product can be written as a *functional* Moyal product, using the language of path integrals over the coordinates $(x(\kappa), y(\kappa))$. For $\kappa = 0$ the surviving coordinate $y(\kappa = 0)$ is predicted to be commutative. As can be seen in (1.3) it corresponds to a specific linear combination of momentum modes. This combination precisely selects the momentum carried by half of the string. As we explain in the text, for zero momentum functionals, the open string vertex treats the momentum carried by half of the string as a commuting coordinate.

Alternatively, we can invert (1.3) and use (1.2) to interpret the Witten star product as the Moyal product in the noncommutative space (x_{2n}, p_{2m-1}) with a nondiagonal form of noncommutative parameter

$$[x_{2n}, p_{2m-1}]_* = i\Theta^{2n,2m-1} \qquad n, m \geq 1\,. \tag{1.4}$$

where the matrix Θ above coincides with one of the matrices relevant in the half-string formalism. In this way Witten's prescription for identifying half strings may be interpreted as identifying the end points of dipoles in the noncommutative space (1.4). It is interesting that p_{2m-1} have the interpretation of coordinates and thus x_{2m-1} should be interpreted as momenta.

We will compare in detail our results with those in the earlier work of Bars [4] finding broad agreement, but some subtle differences. His result, given in terms of Moyal pairs involving the original modes, is indeed compatible with (1.4). The subtleties concerning the invertibility of this matrix are presumably responsible for the absence of the above mentioned commuting coordinate in the framework of [4].

We have also explored the possibility of rewriting open string field theory in the language of NC field theory. For this we simplify the string field theory action by simply ignoring the ghost sector, by working at zero momentum and in a fixed gauge. The kinetic operator is then L_0 and a kernel representing its action on the continuous basis of oscillators is required and seems to exist, albeit in a regulated form. It is then possible to rewrite the L_0 action as star commutation. Our final results are suggestive, but are clearly of very preliminary nature.

This paper is organized as follows. In section 2 we explain how the basic two-dimensional Moyal product can be described as a three-vertex in a Fock state formalism. In section 3, we introduce the continuous basis of oscillators needed to diagonalize the Neumann matrices and rewrite the open string vertex in terms of them. The comparison between sections 2 and 3 is done in section 4 where we identify the Moyal structures in

[2]Strictly speaking, the two products are related by multiplication by an overall (infinite) constant factor; see sections 5 and 6.

the star algebra, and give the physical interpretation for the commuting coordinate. In section 5 we give a functional description of the continuous tensor product of the Moyal algebra. In section 6 we compare our results with those of Bars, and discuss some issues related to the half string formalism. In section 7 we attempt to rewrite open string field theory in the language of noncommutative field theory. We offer some concluding remarks in section 8.

A remark on notation. Although our results can of course be interpreted as defining the Witten string field theory product as the multiplication law of a noncommutative algebra, for clarity we will not take this step, instead writing the noncommutative product explicitly as "$f * g$."

2. Oscillator vertex for the Moyal product

The three string vertex is ordinarily expressed as a quadratic form in oscillators acting on the vacuum. More precisely, we have oscillators from the three state spaces, those of the two input string fields and that of the output product, acting on the tensor product of vacua. In order to understand how to interpret such a vertex as a Moyal product we will calculate here an oscillator-form three-vertex for the Moyal multiplication in two dimensional space.

It is useful first to recall how ordinary commutative pointwise multiplication of functions can be encoded in a three-vertex written in oscillator form. This result was given in [12, eq. (3.15)]:

$$|V_3\rangle = \left(\frac{2}{3\sqrt{\pi}}\right)^{1/2} \exp\left[\frac{1}{6}\left(a_1^\dagger a_1^\dagger + a_2^\dagger a_2^\dagger + a_3^\dagger a_3^\dagger\right) - \frac{2}{3}\left(a_1^\dagger a_2^\dagger + a_2^\dagger a_3^\dagger + a_3^\dagger a_1^\dagger\right)\right]|0\rangle\,, \quad (2.1)$$

where the subscripts one and two on the oscillators refer to the first function and second function to be multiplied respectively, while the subscript three is associated to the product. This result goes along with the definitions

$$\hat{x} = \frac{i}{\sqrt{2}}\left(a - a^\dagger\right), \qquad \hat{p} = \frac{1}{\sqrt{2}}\left(a + a^\dagger\right), \quad (2.2)$$

and the construction of the position states

$$\langle x| = \frac{1}{\pi^{1/4}}\langle 0|\exp\left(-\frac{1}{2}x^2 + \sqrt{2}ia\,x + \frac{1}{2}aa\right). \quad (2.3)$$

Given two functions $f(x) = \langle x|f\rangle$ and $g(x) = \langle x|g\rangle$, then $(f \cdot g)(x) = \langle x|\langle f|\langle g||V_3\rangle$. We are after the generalization of (2.1) for the case of Moyal multiplication. It should be noted from the outset that for a Moyal product we need two coordinates — so the vertex will be a modification of the above involving two sets of oscillators. When the noncommutativity is set to zero, we must recover two copies of the above vertex. We begin with the standard definition of the Moyal product in momentum space for $\mathbb{R}^{2d}$:

$$(f * g)(k) = \int \frac{dk_1}{(2\pi)^{2d}}\frac{dk_2}{(2\pi)^{2d}}(2\pi)^{2d}\delta(k_1 + k_2 - k)e^{-\frac{i}{2}k_{1\mu}\Theta^{\mu\nu}k_{2\nu}}f(k_1)g(k_2)\,. \quad (2.4)$$

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with Fourier transformation definitions

$$h(k) = \int dx e^{ikx} h(x)\,, \quad h(x) = \int \frac{dk}{(2\pi)^{2d}} e^{-ikx} h(k)\,. \tag{2.5}$$

We pass to coordinate space to find a kernel $K(x_1, x_2, x_3)$ defined from

$$(f * g)(x_3) \equiv \int dx_1 dx_2 K(x_1, x_2, x_3) f(x_1) g(x_2)\,. \tag{2.6}$$

Using the formula for momentum space we have:

$$K(x_1, x_2, x_3) = \int \frac{dk_1}{(2\pi)^{2d}} \frac{dk_2}{(2\pi)^{2d}} \exp\left(-\frac{1}{2} k_1 \cdot i\Theta \cdot\, k_2 + k_1 \cdot i(x_1 - x_3) + k_2 \cdot i(x_2 - x_3) \right). \tag{2.7}$$

Here Θ is a $2d \times 2d$ matrix and we use $\cdot$ to denote scalar product, or sum over $2d$ component indices. The integral can be done by completing squares and one finds the standard result

$$K(x_1, x_2, x_3) = \frac{1}{\pi^{2d} \det \Theta} \exp\left(-2i(x_1 - x_3)\Theta^{-1} (x_2 - x_3) \right), \tag{2.8}$$

or, in manifestly cyclic form

$$K(x_1, x_2, x_3) = \frac{1}{\pi^{2d} \det \Theta} \exp\left(-2i \left[x_1 \Theta^{-1} x_2 + x_2 \Theta^{-1} x_3 + x_3 \Theta^{-1} x_1 \right] \right). \tag{2.9}$$

When Θ is skew diagonal, it is possible write the star product in a mixed momentum and coordinate basis and the kernel takes form of a product of δ-functions [16, 4]. More explicitly write

$$\vec{x} = (x^M, y^M)\,, \tag{2.10}$$

where $x^M, y^M \in \mathbb{R}^d$, $M = 1, \cdots d$ and

$$[x^M, y^N] = i\theta^{MN}\,. \tag{2.11}$$

Now consider a mixed basis $(x^M, p^{(y)}_N)$ and define

$$x_L = x^M + \frac{1}{2}\theta^{MN} p^{(y)}_N\,, \qquad x_R = x^M - \frac{1}{2}\theta^{MN} p^{(y)}_N\,. \tag{2.12}$$

Then the Moyal product can be written as

$$(f * g)(x_L, x_R) = \frac{1}{(4\pi)^d \sqrt{\det \Theta}} \int d^d z f(x_L, z) g(z, x_R)\,. \tag{2.13}$$

The above representation has a nice physical interpretation [16] in terms of dipoles interacting by joining their ends together. The center of the dipole is specified by the center of mass coordinates $\vec{x}$. The momenta $\vec{p} = (p^{(x)}_M, p^{(y)}_N)$ defines the extent of the dipole as $\vec{\Delta} = \Theta \cdot \vec{p}$. Then x_L, x_R are coordinates for the left and right end of the dipole.[3]

[3] Since the dipole endpoints are noncommutative, we cannot simultaneously identify the other coordinates (y_L, y_R) when joining the dipoles in the star product.

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Now restrict attention to two dimensional noncommutative space, i.e., set $d = 1$. Write a general cyclic ansatz for a three-vertex, by considering oscillators (a, b) associated to the coordinates x^μ, with $\mu = 1, 2$.

$$(x_1, y_1\,; x_2, y_2\,; x_3, y_3) \leftrightarrow \left(a_1^\dagger, b_1^\dagger\,; a_2^\dagger, b_2^\dagger\,; a_3^\dagger, b_3^\dagger\right). \tag{2.14}$$

The position eigenstates can be defined using (2.2) together with the analogous relations

$$\hat{y} = \frac{i}{\sqrt{2}}\left(b - b^\dagger\right); \qquad \hat{q} = \frac{1}{\sqrt{2}}\left(b + b^\dagger\right). \tag{2.15}$$

Using 2-component notation $\vec{x} = (x, y), \vec{a} = (a, b)$, we have

$$\begin{aligned} \langle \vec{x}| &= \frac{1}{\sqrt{\pi}}\langle 0|\exp\left(-\frac{1}{2}\vec{x}\cdot\vec{x} + \sqrt{2}i\vec{a}\cdot\vec{x} + \frac{1}{2}\vec{a}\cdot\vec{a}\right), \\ \langle \vec{p}| &= \frac{1}{\sqrt{\pi}}\langle 0|\exp\left(-\frac{1}{2}\vec{p}\cdot\vec{p} + \sqrt{2}\vec{a}\cdot\vec{p} - \frac{1}{2}\vec{a}\cdot\vec{a}\right). \end{aligned} \tag{2.16}$$

The three-vertex will be written as

$$|V_3\rangle = N \exp\left(-\frac{1}{2}\vec{A}^\dagger V \vec{A}^\dagger\right)|0\rangle\,, \tag{2.17}$$

where V is a 6×6 matrix, and $\vec{A}$ a six component vector

$$V = \begin{pmatrix} u & v & v^T \\ v^T & u & v \\ v & v^T & u \end{pmatrix}, \qquad \vec{A}^\dagger = (\vec{a}_1^\dagger, \vec{a}_2^\dagger, \vec{a}_3^\dagger)\,. \tag{2.18}$$

with u, v two by two matrices. We now require that

$$K(\vec{x}_1, \vec{x}_2, \vec{x}_3) = (\langle \vec{x}_1| \otimes \langle \vec{x}_2| \otimes \langle \vec{x}_3|)\,|V_3\rangle = \frac{1}{\pi^2 \det\Theta}\exp\left(-\frac{1}{2}\vec{X}K\vec{X}\right), \tag{2.19}$$

where use was made of (2.9), and we have defined

$$K = (2i)\begin{pmatrix} 0 & \Theta^{-1} & -\Theta^{-1} \\ -\Theta^{-1} & 0 & \Theta^{-1} \\ \Theta^{-1} & -\Theta^{-1} & 0 \end{pmatrix}, \qquad \vec{X} = (\vec{x}_1, \vec{x}_2, \vec{x}_3)\,. \tag{2.20}$$

In this notation, (2.16) allows one to write

$$\langle X| = \langle \vec{x}_1| \otimes \langle \vec{x}_2| \otimes \langle \vec{x}_3| = \frac{1}{\pi\sqrt{\pi}}\langle 0|\exp\left(-\frac{1}{2}\vec{X}\cdot\vec{X} + \sqrt{2}i\vec{A}\cdot\vec{X} + \frac{1}{2}\vec{A}\cdot\vec{A}\right). \tag{2.21}$$

Substituting back into the main condition (2.19) and doing the contraction we find:

$$\frac{1}{\pi^2\det\Theta}\exp\left(-\frac{1}{2}\vec{X}K\vec{X}\right) = \frac{N}{\pi\sqrt{\pi}}\frac{1}{\sqrt{\det(1+V)}}\exp\left(-\frac{1}{2}\vec{X}\,\frac{1-V}{1+V}\vec{X}\right), \tag{2.22}$$

and conclude that

$$V = \frac{1-K}{1+K}\,, \qquad N = \frac{\sqrt{\det(1+V)}}{\sqrt{\pi}\det\Theta}\,. \tag{2.23}$$

We now take $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$, and find that for $\theta \neq 0$ the matrix $(1+K)$ appearing in V is invertible. The resulting expression for V in terms of the u, v matrices in (2.18) is

$$u = \frac{-4+\theta^2}{12+\theta^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad v = \frac{4}{12+\theta^2}\begin{pmatrix} 2 & i\theta \\ -i\theta & 2 \end{pmatrix}. \tag{2.24}$$

Moreover

$$N = \frac{8}{\sqrt{\pi}(12+\theta^2)} = \frac{2}{3\sqrt{\pi}}\frac{1}{(1+\theta^2/12)}. \tag{2.25}$$

All in all, the form of the vertex is

$$\begin{aligned} |V_3(\theta)\rangle = \frac{2}{3\sqrt{\pi}}\frac{1}{1+\theta^2/12}\exp\Bigg[&-\frac{1}{2}\left(\frac{-4+\theta^2}{12+\theta^2}\right)(a_1^\dagger a_1^\dagger + b_1^\dagger b_1^\dagger + \text{cyclic}) - \\ &-\left(\frac{8}{12+\theta^2}\right)(a_1^\dagger a_2^\dagger + b_1^\dagger b_2^\dagger + \text{cyclic}) - \\ &-\left(\frac{4i\theta}{12+\theta^2}\right)(a_1^\dagger b_2^\dagger - b_1^\dagger a_2^\dagger + \text{cyclic})\Bigg]|0\rangle. \end{aligned} \tag{2.26}$$

This is the desired form of the vertex for canonical Moyal. The limit $\theta \to 0$ is smooth and gives:

$$|V_3(\theta=0)\rangle = \frac{2}{3\sqrt{\pi}}\exp\left[\frac{1}{6}\left(a_1^\dagger a_1^\dagger + b_1^\dagger b_1^\dagger + \text{cyclic}\right) - \frac{2}{3}\left(a_1^\dagger a_2^\dagger + b_1^\dagger b_2^\dagger + \text{cyclic}\right)\right]|0\rangle. \tag{2.27}$$

Happily, this agrees with the commutative result reviewed in (2.1).

3. Open string star in the continuous oscillator basis

In this section we will examine the three string vertex and use the recent diagonalization of the Neumann matrices to rewrite the vertex in a basis of oscillators with a continuous mode label. We will then perform a redefinition of the oscillators in such a way to allow a comparison with the Moyal form of the vertex determined in the previous section.

We begin by recalling the main results of [9] that we will need. We have the set of eigenvectors of $v_m(\kappa)$ of matrices M^{rs}_{mn} which are labeled by a continuous parameter $-\infty < \kappa < \infty$, i.e.

$$\sum_{n=1}^{\infty} M^{rs}_{mn} v_n(\kappa) = \mu^{rs}(\kappa) v_m(\kappa), \tag{3.1}$$

where the eigenvalues are given by

$$\begin{aligned} \mu(\kappa) = \mu^{11}(\kappa) &= -\frac{1}{1+2\cosh\frac{\pi\kappa}{2}}, \\ \mu^{12}(\kappa) &= \frac{1+\cosh\frac{\pi\kappa}{2}+\sinh\frac{\pi\kappa}{2}}{1+2\cosh\frac{\pi\kappa}{2}}, \\ \mu^{21}(\kappa) &= \frac{1+\cosh\frac{\pi\kappa}{2}-\sinh\frac{\pi\kappa}{2}}{1+2\cosh\frac{\pi\kappa}{2}}, \end{aligned} \tag{3.2}$$

and they satify the relations

$$\mu + \mu^{12} + \mu^{21} = 1, \qquad \mu^{12}\mu^{21} = \mu(\mu - 1)\,. \tag{3.3}$$

The eigenvector $v_n(\kappa)$ is given by the generating function:[4]

$$f_\kappa(z) = \sum_{n=1}^{\infty} \frac{v_n(\kappa)}{\sqrt{n}} z^n = \frac{1}{N(\kappa)^{1/2}} \frac{1}{\kappa} \left(1 - e^{-\kappa \tan^{-1} z}\right), \tag{3.4}$$

where $N(\kappa)$ is given by [13]

$$N(\kappa) = \frac{2}{\kappa} \sinh \frac{\pi\kappa}{2}\,. \tag{3.5}$$

Note also that

$$\mu^{rs}(-\kappa) = \mu^{sr}(\kappa)\,, \tag{3.6}$$

where we have defined $\mu^{r+1,s+1} = \mu^{rs}$ for superscripts mod 3. The twist matrix C has a well-defined action on the eigenvectors

$$\sum_{n=1}^{\infty} C_{mn} v_n(\kappa) = -v_m(-\kappa)\,, \qquad C_{mn} = (-1)^m \delta_{mn}. \tag{3.7}$$

It follows from this equation that the even and odd components of the eigenvectors satisfy the relations

$$v_{2n+1}(-\kappa) = v_{2n+1}(\kappa)\,, \qquad v_{2n}(-\kappa) = -v_{2n}(\kappa)\,. \tag{3.8}$$

The eigenfunctions can be shown to be orthogonal and complete,

$$\begin{aligned} \sum_{n=1}^{\infty} v_n(\kappa_1) v_n(\kappa_2) \,&=\, \delta(\kappa_1 - \kappa_2)\,, \\ \int_{-\infty}^{\infty} d\kappa v_m(\kappa) v_n(\kappa) \,&=\, \delta_{m,n}\,. \end{aligned} \tag{3.9}$$

Due to the relations (3.8), we can separate the even and odd modes and write the completeness and orthogonality relations (3.9)

$$\begin{aligned} 2\sum_{n=1}^{\infty} v_{2n-1}(\kappa_1) v_{2n-1}(\kappa_2) \,&=\, \delta(\kappa_1 - \kappa_2)\,, \qquad & 2\sum_{n=1}^{\infty} v_{2n}(\kappa_1) v_{2n}(\kappa_2) \,&=\, \delta(\kappa_1 - \kappa_2)\,, \\ 2\int_0^{\infty} d\kappa v_{2m}(\kappa) v_{2n}(\kappa) \,&=\, \delta_{m,n}, \qquad & 2\int_0^{\infty} d\kappa v_{2m-1}(\kappa) v_{2n-1}(\kappa) \,&=\, \delta_{m,n}\,. \end{aligned} \tag{3.10}$$

[4]The v_n's here differ from those in [9] by the inclusion of the normalization factor $N(\kappa)^{-1/2}$.

We stress that these equations hold only for $\kappa_1, \kappa_2 > 0$. In terms of odd and even modes equation (3.1) becomes

$$\sum_{m=1}^{\infty} M^{rs}_{2n,2m} v_{2m}(\kappa) = \frac{1}{2}\left(\mu^{rs}(\kappa) + \mu^{sr}(\kappa)\right) v_{2n}(\kappa), \tag{3.11}$$

$$\sum_{m=1}^{\infty} M^{rs}_{2n,2m-1} v_{2m-1}(\kappa) = \frac{1}{2}\left(\mu^{rs}(\kappa) - \mu^{sr}(\kappa)\right) v_{2n}(\kappa), \tag{3.12}$$

$$\sum_{m=1}^{\infty} M^{rs}_{2n-1,2m-1} v_{2m-1}(\kappa) = \frac{1}{2}\left(\mu^{rs}(\kappa) + \mu^{sr}(\kappa)\right) v_{2n-1}(\kappa), \tag{3.13}$$

$$\sum_{m=1}^{\infty} M^{rs}_{2n-1,2m} v_{2m}(\kappa) = \frac{1}{2}\left(\mu^{rs}(\kappa) - \mu^{sr}(\kappa)\right) v_{2n-1}(\kappa)\ . \tag{3.14}$$

These properties allow us to introduce new oscillators whose mode number is a continuous parameter. From (3.10), it is convenient to introduce new continuous oscillators e_κ and o_κ associated to the even and odd mode sums respectively:

$$o^\dagger_\kappa = -\sqrt{2}\, i \sum_{n=1}^{\infty} v_{2n-1}(\kappa) a^\dagger_{2n-1}\,, \qquad e^\dagger_\kappa = \sqrt{2} \sum_{n=1}^{\infty} v_{2n}(\kappa)\, a^\dagger_{2n}\,, \tag{3.15}$$

$$a^\dagger_{2n} = \sqrt{2} \int_0^\infty d\kappa v_{2n}(\kappa) e^\dagger_\kappa\,, \qquad a^\dagger_{2n-1} = \sqrt{2} i \int_0^\infty d\kappa v_{2n-1}(\kappa) o^\dagger_\kappa\,. \tag{3.16}$$

We have introduced a factor of i in the first equation so that they have the same BPZ conjugation property

$$\mathrm{bpz}(o_\kappa) = -o^\dagger_\kappa\,, \qquad \mathrm{bpz}(e_\kappa) = -e^\dagger_\kappa\,. \tag{3.17}$$

The new oscillators satisfy the commutation relations

$$\left[o_\kappa, o^\dagger_{\kappa'}\right] = \left[e_\kappa, e^\dagger_{\kappa'}\right] = \delta\left(\kappa - \kappa'\right), \qquad \left[o_\kappa, e^\dagger_{\kappa'}\right] = \left[e_\kappa, o^\dagger_{\kappa'}\right] = 0\,. \tag{3.18}$$

Note that for $\kappa = 0$, the $e_{\kappa=0}$ oscillator vanishes, and we only have $o_{\kappa=0}$. This is because the $\kappa = 0$ eigenvector is C-odd. Note also that the change of basis from the discrete oscillators into the continuous one (3.15) is a unitary transformation, as can be checked using (3.10).

Using equations (3.11)–(3.16) and the completeness relations (3.10), after some algebra the three-string vertex

$$|V_3\rangle = \exp\left[-\frac{1}{2}\sum_{r,s}\sum_{m,n} a^{r\dagger}_m (CM^{rs})_{mn} a^{s\dagger}_n\right]|0\rangle\,, \tag{3.19}$$

can then be written in terms of o_κ, e_κ basis as

$$|V_3\rangle = \exp\Bigg[\sum_{r,s}\int_0^\infty d\kappa \Bigg\{ -\frac{1}{4}\left(\mu^{rs}(\kappa) + \mu^{sr}(\kappa)\right)\left(e^{(r)}_\kappa{}^\dagger e^{(s)}_\kappa{}^\dagger + o^{(r)}_\kappa{}^\dagger o^{(s)}_\kappa{}^\dagger\right) + \frac{i}{4}(\mu^{rs} - \mu^{sr})\left(o^{(r)}_\kappa{}^\dagger e^{(s)}_\kappa{}^\dagger - e^{(r)}_\kappa{}^\dagger o^{(s)}_\kappa{}^\dagger\right)\Bigg\}\Bigg]|0\rangle\,, \tag{3.20}$$

where $|0\rangle$ is now interpreted as a "continuous tensor product" of oscillator ground states for e_κ, o_κ. More explicitly, (3.20) can be written as

$$
\begin{aligned}
|V_3\rangle = \exp\Bigg[\int_0^\infty d\kappa \Bigg\{ & -\frac{1}{2}\mu(\kappa)\left(o_\kappa^{(1)\dagger} o_\kappa^{(1)\dagger} + e_\kappa^{(1)\dagger} e_\kappa^{(1)\dagger} + \text{cyc}\right) - \\
& - \frac{1}{2}\left(\mu^{12}(\kappa) + \mu^{21}(\kappa)\right)\left(o_\kappa^{(1)\dagger} o_\kappa^{(2)\dagger} + e_\kappa^{(1)\dagger} e_\kappa^{(2)\dagger} + \text{cyc}\right) - \\
& - \frac{i}{2}\left(\mu^{12}(\kappa) - \mu^{21}(\kappa)\right)\left(e_\kappa^{(1)\dagger} o_\kappa^{(2)\dagger} - o_\kappa^{(1)\dagger} e_\kappa^{(2)\dagger} + \text{cyc}\right)\Bigg\}\Bigg]|0\rangle\,. \quad (3.21)
\end{aligned}
$$

This is the desired form of the open string three-vertex. In the next section we identify the Moyal structures explicitly. The above matter vertex is restricted to the $p = 0$ sector, and is appropriately normalized.

4. Identification of Moyal structures

In this section we use the results from the two previous sections to show that the star product indeed corresponds to Moyal products for a continuous set of variables parametrized by $\kappa \in [0, \infty)$. We identify these variables and also give an interpretation for the commuting mode encountered at $\kappa = 0$.

We have now completed our preliminary work finding both the oscillator representation of the Moyal product in section 2, eq. (2.26), and a presentation (3.21) of the three-string vertex as a continuous tensor product of three-vertices. Comparing the exponents in these vertices we see that an identification

$$(a^\dagger, b^\dagger) \leftrightarrow (e_\kappa^\dagger, o_\kappa^\dagger)\,, \quad (4.1)$$

requires the conditions

$$\mu(\kappa) = \frac{-4+\theta^2}{12+\theta^2}\,, \qquad \mu^{12}(\kappa) + \mu^{21}(\kappa) = \frac{16}{12+\theta^2}\,, \qquad \mu^{12}(\kappa) - \mu^{21}(\kappa) = \frac{8\theta}{12+\theta^2}\,. \quad (4.2)$$

Note that the consistency condition $\mu + \mu^{12} + \mu^{21} = 1$ is satisfied. Making use of the explicit expressions (3.2) we find that the above equations are all satisfied by choosing

$$\theta(\kappa) = 2\tanh\left(\frac{\pi\kappa}{4}\right)\,. \quad (4.3)$$

This is the noncommutativity parameter associated to the Moyal algebras. Note in particular that for $\kappa = 0$ where the pair of oscillators collapses to just $o_{\kappa=0}$ we have a commutative product. The noncommutativity parameter grows as κ grows but is bounded: $\theta(\kappa) \leq 2$.

Note that the Witten vertex differs from the Moyal vertex by a c-number prefactor, as can be seen by comparing (2.26) with (3.21). This means that, strictly speaking, the open string field theory product $*_W$ and the Moyal product $*$ are related by an overall (infinite) constant factor,

$$f *_W g = C' f * g\,, \quad (4.4)$$

which we will write out later in equation (5.24). Such a factor can of course be eliminated by a redefinition of the string field $f \to C' f$.

Having identified the noncommutative parameter we must now determine what are the explicit forms of the coordinates that enter the Moyal product. It is natural to expect that those will be continuous coordinate modes $x(\kappa)$ associated to the usual position modes x_n by relations similar to those in (3.15). As hinted at before, we will have to use even and odd modes, and a Fourier transformation of the odd modes will be necessary.

For this purpose we consider the position and momentum eigenstates

$$\begin{aligned}\langle\vec{X}(\sigma)| &= \langle 0|\exp\left(-\vec{x}\cdot E^{-2}\cdot\vec{x}+2i\vec{a}\cdot E^{-1}\cdot\vec{x}+\frac{1}{2}\vec{a}\cdot\vec{a}\right),\\ \langle\vec{P}(\sigma)| &= \langle 0|\exp\left(-\frac{1}{4}\vec{p}\cdot E^{2}\cdot\vec{p}+\vec{a}\cdot E\cdot\vec{p}-\frac{1}{2}\vec{a}\cdot\vec{a}\right),\end{aligned}\tag{4.5}$$

where

$$\hat{x}=\frac{i}{2}E\cdot\left(a-a^\dagger\right),\qquad \hat{p}=E^{-1}\cdot\left(a+a^\dagger\right),\qquad E_{mn}=\sqrt{\frac{2}{n}}\delta_{mn}\,.\tag{4.6}$$

(This is working in the $\alpha'=1/2$ convention). More explicitly

$$\hat{x}_n=\frac{i}{\sqrt{2n}}\cdot\left(a_n-a_n^\dagger\right),\qquad \hat{p}_n=\sqrt{\frac{n}{2}}\left(a_n+a_n^\dagger\right),\qquad n\geq 1\,,\tag{4.7}$$

and the zero mode expressions are those in (2.2). The above go along with the expansions

$$\widehat{X}(\sigma)=\hat{x}_0+\sqrt{2}\sum_{n=1}^{\infty}\hat{x}_n\cos n\sigma\,,\qquad \pi\widehat{P}(\sigma)=\hat{p}_0+\sqrt{2}\sum_{n=1}^{\infty}\hat{p}_n\cos n\sigma\,.\tag{4.8}$$

We must now define new coordinate and momentum operators associated to the oscillators $(e_\kappa,e_\kappa^\dagger)$ and $(o_\kappa,o_\kappa^\dagger)$ we have introduced. We let

$$(\hat{x}_\kappa,\hat{q}_\kappa)\leftrightarrow(e_\kappa,e_\kappa^\dagger)\,,\qquad (\hat{y}_\kappa,\hat{l}_\kappa)\leftrightarrow(o_\kappa,o_\kappa^\dagger)\,,\tag{4.9}$$

using the standard correspondences for zero-modes (2.2):

$$\begin{aligned}\hat{x}_\kappa &= \frac{i}{\sqrt{2}}\left(e_\kappa-e_\kappa^\dagger\right), & \hat{q}_\kappa &= \tfrac{1}{\sqrt{2}}\left(e_\kappa+e_\kappa^\dagger\right),\\ \hat{y}_\kappa &= \frac{i}{\sqrt{2}}\left(o_\kappa-o_\kappa^\dagger\right), & \hat{l}_\kappa &= \frac{1}{\sqrt{2}}\left(o_\kappa+o_\kappa^\dagger\right).\end{aligned}\tag{4.10}$$

The coordinate operators $\hat{x}_\kappa$ and $\hat{y}_\kappa$ are the ones for which the 3-string vertex has been put in Moyal form. We must therefore now express them in terms of the original operators $(\hat{x}_n,\hat{p}_n)$. For this we use (3.15) to first pass to $(a_n,a_n^\dagger)$ oscillators and then (4.7) to pass to $(\hat{x}_n,\hat{p}_n)$ operators. One immediately finds

$$\hat{x}_\kappa = \sqrt{2}\sum_{n=1}^{\infty}v_{2n}(\kappa)\sqrt{2n}\,\hat{x}_{2n}\,,\tag{4.11}$$

$$\hat{y}_\kappa = -\sqrt{2}\sum_{n=1}^{\infty}\frac{v_{2n-1}(\kappa)}{\sqrt{2n-1}}\hat{p}_{2n-1}\,,\tag{4.12}$$

$$\hat{q}_\kappa = \sqrt{2}\sum_{n=1}^{\infty}\frac{v_{2n}(\kappa)}{\sqrt{2n}}\hat{p}_{2n}\,,\tag{4.13}$$

$$\hat{l}_\kappa = \sqrt{2}\sum_{n=1}^{\infty}v_{2n-1}(\kappa)\sqrt{2n-1}\,\hat{x}_{2n-1}\,.\tag{4.14}$$

Here we see that the Moyal coordinates, the eigenvalues of $\hat{x}_\kappa$ and $\hat{y}_\kappa$, are respectively (i) linear combination of even conventional coordinates, and (ii) linear combinations of odd momenta. Thus the Moyal structure is canonical for the multiplication of string functionals where the odd coordinates x_{2n-1} are replaced by odd momenta p_{2n-1} via Fourier transformation. In other words given string functionals $\Psi_i(\{x_{2n}\},\{x_{2n-1}\})$ to be star multiplied, use of Moyal product requires the transformations

$$\Psi_i(\{x_{2n}\},\{x_{2n-1}\}) \to \widetilde{\Psi}_i(\{x_{2n}\},\{p_{2n-1}\}) \to \Psi_i^M(x(\kappa),y(\kappa))\,, \tag{4.15}$$

where $x(\kappa)$ and $y(\kappa)$ denote the eigenvalues of $\hat{x}_\kappa$ and $\hat{y}_\kappa$ respectively. The first arrow above denotes Fourier transformation, and in the second arrow we just reexpress the coordinate and momenta in terms of continuous variables using (4.11) and (4.12). In this final form, with superscript M for Moyal, the star product is just canonical Moyal product with $\theta(\kappa)$ for each $\kappa \geq 0$, in a way that will be made more precise in the next subsection.

It is of interest to identify the nature of the commuting coordinate associated to $\kappa = 0 \to \theta(\kappa) = 0$. Indeed for $\kappa = 0$ the eigenvector $v(\kappa = 0)$ only has odd components, and therefore, from (4.11) and (4.12) only $\hat{y}_{\kappa=0}$ survives. This is explicitly

$$\hat{y}_{\kappa=0} = -\sqrt{2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\hat{p}_{2n-1} = -\sqrt{2}\left(\hat{p}_1 - \frac{1}{3}\hat{p}_3 + \frac{1}{5}\hat{p}_5 - \cdots\right). \tag{4.16}$$

We can identify this right hand side using (4.8). We note that the above linear combination is, with $p_0 = 0$, just the momentum carried by half the string

$$\hat{y}_{\kappa=0} = -\sqrt{2}\int_0^{\pi/2}\pi\hat{P}(\sigma)d\sigma = -\sqrt{2}\pi\hat{P}_L\,. \tag{4.17}$$

Our result therefore states that $\hat{P}_L$ must behave as an ordinary commuting coordinate as far as the string field vertex is concerned. Indeed, this is the case for zero momentum functionals, as we explain now. Since the open string vertex is defined by gluing right half strings to left half strings it implements the following conditions when multiplying string (1) times string (2) to give string (3):

$$\hat{P}_R^{(3)} = -\hat{P}_L^{(1)}\,, \qquad \hat{P}_R^{(1)} = -\hat{P}_L^{(2)}\,, \qquad \hat{P}_R^{(2)} = -\hat{P}_L^{(2)}\,. \tag{4.18}$$

But for zero momentum string states $\hat{P}_R^{(1)} = -\hat{P}_L^{(1)}$ and $\hat{P}_R^{(2)} = -\hat{P}_L^{(2)}$, and as a consequence we have that the vertex requires

$$\hat{P}_L^{(1)} = \hat{P}_L^{(2)} = \hat{P}_L^{(3)}\,, \tag{4.19}$$

which is the statement that $\hat{P}_L$ eigenvalues behave as a commuting *coordinate* at the string vertex.

It is interesting to note that for zero momentum functionals $\hat{P}_L$ is, up to a constant, the same as the operator $\hat{P}_L - \hat{P}_R$. The connection of the $\kappa = 0$ eigenvector to $\hat{P}_L - \hat{P}_R$ is clear from the observation of [12] that this eigenvector implies that the sliver functional is invariant under opposite rigid displacements of the two halves of the string. More precisely, the zero-momentum sliver is annihilated by the action of $\hat{P}_L - \hat{P}_R$. It follows that the commuting coordinate $\hat{y}_{\kappa=0} \sim \hat{P}_L$ vanishes on the sliver. Of course, it will not vanish on general zero-momentum functionals.

5. Open string star as continuous Moyal products

We have shown that the noncommutative algebra of open string field theory can be identified with a continuous tensor product of Heisenberg algebras labeled by $\kappa > 0$, with a noncommutative parameter given in (4.3):

$$[x(\kappa), y(\kappa')]_* = i2\tanh\frac{\pi\kappa}{4}\delta(\kappa-\kappa')\,. \tag{5.1}$$

In this section we shall give a precise description of this continuous tensor product in terms of a functional integral defined on the half line $\kappa > 0$.

We begin our analysis by looking at the inner product in string field theory, and how it would look in the continuous bases. Formally under the procedure outlined in (4.15), the inner product in string field theory becomes

$$\begin{aligned}\int \Psi_1 * \Psi_2 &= \int Dx(\sigma)\Psi_1(x(\sigma))\Psi_2(x(\pi-\sigma)) \\ &= \int \left(\prod_{n=0}^{\infty} Dx_n\right)\Psi_1(\{x_{2n}\},\{x_{2n-1}\})\Psi_2(\{x_{2n}\},\{-x_{2n-1}\}) \\ &= \int \left(\prod_{n=0}^{\infty} Dx_{2n}\right)\left(\prod_{m=1}^{\infty} Dp_{2m-1}\right)\tilde{\Psi}_1(\{x_{2n}\},\{p_{2m-1}\})\,\tilde{\Psi}_2(\{x_{2n}\},\{p_{2m-1}\}) \\ &= \int Dx(\kappa)Dy(\kappa)\Psi_1^M\left[x(\kappa),y(\kappa)\right]\,\Psi_2^M\left[x(\kappa),y(\kappa)\right]\,, \end{aligned} \tag{5.2}$$

where in the third line we have performed a Fourier transform for the odd coordinates, and in the last line we have passed to the continuous variables representing the Moyal coordinates. Thus we see that the integration in string field theory indeed reduces to the standard integration in the Moyal coordinates. Note that the "$-$" sign before x_{2n-1} in Ψ_2 in the second line is important for this identification. This result was foreordained as there is no other integral which respects cyclicity of the trace.

The above formal expressions, however, are not really meaningful until we have specified the precise integral measures Dx_n, Dp_n and $Dx(\kappa), Dy(\kappa)$ used in each step. In particular, it is desirable to have the measures so that the transformations (4.11) and (4.12) are orthogonal transformations. Since the non-commutativity parameter is a tensor under changes of coordinates, as long as we restrict ourselves to orthogonal transformations it is meaningful to speak about the magnitude of the Moyal parameter.

We now define the integral measure in each line of (5.2) by requiring the perturbative string ground state

$$\Psi_{\text{ground}} = \exp\left[-\frac{1}{2}\sum_{n=1}^{\infty} n x_n^2\right], \tag{5.3}$$

has unit norm in each step. Defined this way the transformations (4.11) and (4.12) are then guaranteed to be orthogonal. More explicitly, the measure Dx_n in (5.2) is given by

$$Dx_n = \left(\frac{n}{\pi}\right)^{1/2} dx_n\,. \tag{5.4}$$

This is the formal measure for the metric

$$\|\delta X\|^2 \sim \int_0^\pi \delta X(\sigma)\sqrt{-\frac{d^2}{d\sigma^2}}\delta X(\sigma)\,. \tag{5.5}$$

By choosing our Fourier transform conventions,[5] the measure Dp_n is given by

$$Dp_n = \left(\frac{1}{\pi n}\right)^{1/2} dp_n\ . \tag{5.6}$$

Equivalently we can determine (5.4) and (5.6) by using the inner products of (4.5), e.g. one finds that

$$\langle x'_n|x_n\rangle = \left(\frac{\pi}{n}\right)^{1/2}\delta(x_n - x'_n)\,.$$

Similarly, we define the measure $Dx(\kappa), Dy(\kappa)$ so that

$$\begin{aligned}\int Dx(\kappa)Dy(\kappa)\exp\Big[-\int_0^\infty d\kappa\,\left(x(\kappa)^2+y(\kappa)^2\right)+2i\int_0^\infty d\kappa\,[j_1(\kappa)x(\kappa)+j_2(\kappa)y(\kappa)]\Big] &= \\ = \exp\Big[-\int_0^\infty d\kappa\,\left(j_1(\kappa)^2+j_2(\kappa)^2\right)\Big]&. \end{aligned}\tag{5.7}$$

Formally this is the measure associated to the standard L^2 metric

$$\|(\delta x,\delta y)\|^2 \sim \int_0^\infty \left[(\delta x(\kappa))^2 + (\delta y(\kappa))^2\right]\ . \tag{5.8}$$

An additional point will be necessary. Consider the integral

$$\int Dx(\kappa)Dy(\kappa)\exp\Big[-\int_0^\infty d\kappa G(\kappa)\left(x(\kappa)^2+y(\kappa)^2\right)\Big] = (\det G)^{-1/2}\,. \tag{5.9}$$

A natural way to define the above determinant is to write

$$\ln\det G = \mathrm{Tr}\log\hat{G} = \int_0^\infty d\mu(\kappa)\langle\kappa|\log\hat{G}|\kappa\rangle = \int_0^\infty d\mu(\kappa)\log G(\kappa)\,, \tag{5.10}$$

where the eigenvalues of the operator $\hat{G}$ are given by $G(\kappa)$, and $d\mu(\kappa)$ is the spectral measure. As discussed in [9, 10],

$$d\mu(\kappa) = \frac{\log L}{2\pi}d\kappa,$$

where L is the "level regulator" (i.e. one regulates by approximating K_1 with an $L\times L$ matrix). Thus

$$\int Dx(\kappa)Dy(\kappa)\exp\Big[-\int_0^\infty d\kappa G(\kappa)\left(x(\kappa)^2+y(\kappa)^2\right)\Big] = \exp\left(-\frac{1}{2}\frac{\log L}{2\pi}\int_0^\infty d\kappa\log G(\kappa)\right). \tag{5.11}$$

[5]We use $\langle p|x\rangle = \frac{1}{\sqrt{2}}e^{ipx}$.

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In order to define the functional Moyal product necessary to describe open string star products, we now look into the definition of wave functionals of the Moyal coordinates. The ground state wave function (5.3) can be written in terms of $(x(\kappa), y(\kappa))$ using the inverse relations of (4.11) and (4.12). The result is

$$\Psi^M_{\text{ground}}(x(\kappa), y(\kappa)) = \exp\left[-\frac{1}{2}\int_0^\infty d\kappa\, \left(x(\kappa)^2 + y(\kappa)^2\right)\right]. \tag{5.12}$$

Note that this wave functional indeed has unit normalization, as follows directly from (5.7). We can do this more systematically for general wavefunctionals by introducing

$$\vec{X}(\kappa) = (x(\kappa), y(\kappa)), \qquad \vec{A}_\kappa = (e_\kappa, o_\kappa), \qquad \Theta(\kappa) = \begin{pmatrix} 0 & \theta(\kappa) \\ -\theta(\kappa) & 0 \end{pmatrix} \tag{5.13}$$

with $\theta(\kappa)$ given by (4.3). The position eigenstate $\langle x(\kappa), y(\kappa)|$ is just the state $\langle X|$ in (4.5) restricted to the even modes, times the state $\langle P|$ also in (4.5), restricted to the odd modes, expressed in terms of $(x(\kappa), y(\kappa))$ and (e_κ, o_κ) using equations (4.11) and (4.12) and (3.15). The result is

$$\langle x(\kappa), y(\kappa)| = \langle 0| \exp\left(-\int_0^\infty d\kappa \left[\frac{1}{2}\vec{X}(\kappa)\cdot\vec{X}(\kappa) - \sqrt{2}i\vec{A}_\kappa\cdot\vec{X}(\kappa) - \frac{1}{2}\vec{A}_\kappa\cdot\vec{A}_\kappa\right]\right). \tag{5.14}$$

Wavefunctions associated to states are then easily obtained performing the contraction

$$\Psi^M(x(\kappa), y(\kappa)) = \langle x(\kappa), y(\kappa)|\Psi\rangle\,. \tag{5.15}$$

Note that indeed for the vacuum we recover (5.12). It is easy to see that with the measure introduced in (5.7) we have

$$\int Dx(\kappa)Dy(\kappa)\ |x(\kappa), y(\kappa)\rangle\langle x(\kappa), y(\kappa)| = 1\,. \tag{5.16}$$

As a warmup exercise, and a test of the definitions, we calculate the wave function in Moyal coordinates for the star product of the ground state with itself. Making use of (3.21), (5.14) and (5.11) we find:

$$\begin{aligned}
\Psi^M_{|0\rangle *_W |0\rangle}(x(\kappa), y(\kappa)) &= \langle x(\kappa), y(\kappa)|0\rangle *_W |0\rangle \\
&= \langle x(\kappa), y(\kappa)| \exp\left[-\frac{1}{2}\int_0^\infty \mu(\kappa)\left(o^\dagger_\kappa o^\dagger_\kappa + e^\dagger_\kappa e^\dagger_\kappa\right)\right] \\
&= \exp\left[-\frac{1}{2}\int_0^\infty d\kappa\left(1 + \operatorname{sech}\frac{\pi\kappa}{2}\right)\left(x(\kappa)^2 + y(\kappa)^2\right)\right] \times \\
&\quad \times \exp\left[\frac{\log L}{2\pi}\int_0^\infty d\kappa \log\left(1 + \frac{1}{2}\operatorname{sech}\frac{\pi\kappa}{2}\right)\right].
\end{aligned} \tag{5.17}$$

Although the singular factor in this wave functional may seem strange, it diverges as $L \to \infty$ and the integral is finite, it would also be present in the standard coordinates, as it would arise from the determinant $\det(1 - CM)$. The norm of the above state is

$$\exp\left[-\frac{\log L}{2\pi}\int_0^\infty d\kappa\, \log\left(\frac{1 + \operatorname{sech}\frac{\pi\kappa}{2}}{(1 + \frac{1}{2}\operatorname{sech}\frac{\pi\kappa}{2})^2}\right)\right]. \tag{5.18}$$

On the other hand, the norm of $|0\rangle *_W |0\rangle$ computed in the oscillator basis is given by $(\det(1-M^2))^{-1/2}$ which is precisely eq (5.18) if we use (5.10) and (3.2).

With the properly defined measures we can now write the continuous Moyal star product in terms of the functional form. Using

$$\langle x(\kappa), y(\kappa)|\Psi_1 *_W \Psi_2\rangle = \langle x(\kappa), y(\kappa)|\langle\Psi_1|\langle\Psi_2|V_3\rangle\,, \tag{5.19}$$

and introducing two complete sets of coordinates via (5.16), we can write

$$\begin{aligned}(\Psi_1^M *_W \Psi_2^M)\,(x(\kappa), y(\kappa)) = \int \; & Dx_1(\kappa)Dy_1(\kappa)Dx_2(\kappa)Dy_2(\kappa)\,\times \\ & \times K\left(\vec{X}_1(\kappa), \vec{X}_2(\kappa), \vec{X}(\kappa)\right)\times \\ & \times \Psi_1^M\,(x_1(\kappa), y_1(\kappa))\;\Psi_2^M\,(x_2(\kappa), y_2(\kappa))\,,\end{aligned} \tag{5.20}$$

where the kernel K is given by

$$K\left(\vec{X}_1, \vec{X}_2, \vec{X}_3\right) = \left(\langle\vec{X}_1|\otimes\langle\vec{X}_2|\otimes\langle\vec{X}_3|\right)|V_3\rangle\,. \tag{5.21}$$

With $\langle\vec{X}|$ given by (5.14) and $|V_3\rangle$ by (3.21), a calculation gives:

$$K\left(\vec{X}_1, \vec{X}_2, \vec{X}_3\right) = \exp\left[-2i\int_0^\infty d\kappa\left(\vec{X}_1(\kappa)-\vec{X}_3(\kappa)\right)\cdot\Theta^{-1}(\kappa)\cdot\left(\vec{X}_2(\kappa)-\vec{X}_3(\kappa)\right)+\frac{\log L}{2\pi}C\right], \tag{5.22}$$

where C is a constant given by

$$C = D\int_0^\infty d\kappa\log\left[\frac{1}{8}\left(\frac{12+\theta^2(\kappa)}{\theta^2(\kappa)}\right)\right] = D\int_0^\infty d\kappa\,\log\left[\frac{1}{8}\left(1+3\left(\coth\frac{\pi\kappa}{4}\right)^2\right)\right], \tag{5.23}$$

where D is the number of space-time dimensions. The two equations above, together with (5.20), give the explicit reformulation of the open string star product as a functional Moyal product. Our work has been precise enough to keep track of infinite factors that appear in this transcription and that are well-known to arise in the usual discrete formulations.

Note that constant C obtained above is different from that constant which would arise by exponentiating the prefactor in the standard definition (2.8) of the Moyal kernel — this is due to the difference in normalization between (2.26) and (3.21). The ratio of these two constants is the constant C' in (4.4) defining the overall multiplicative factor relating the Witten and Moyal products:

$$C' = \exp\left(\frac{\log L}{2\pi}D\int_0^\infty d\kappa\log\left[\frac{1}{8}(12+\theta^2(\kappa))\right]\right). \tag{5.24}$$

In the full theory, the factors C and C' will also obtain ghost contributions, possibly changing D to $D-26$.

6. Relation with half-string basis and Bars' work

In this section we discuss the relation of the κ-basis to the half string basis and compare our result with that of Bars [4].

The open string star product can be written in a functional form in terms of overlap of half strings

$$(\Psi_1 * \Psi_2)\,[x_L(\sigma), x_R(\sigma)] \equiv \int \prod_{0\le\sigma\le\frac{\pi}{2}} Dz(\sigma)\Psi_1[x_L(\sigma), z(\sigma)]\Psi_2[z(\sigma), x_R(\sigma)]\,, \tag{6.1}$$

where the open string halves are expanded as:

$$x_L(\sigma) = x(\sigma) = l_0 + \sqrt{2}\sum_{m=1}^{\infty} l_{2m}\cos(2m\sigma), \qquad 0 \le \sigma \le \frac{\pi}{2}\,, \tag{6.2}$$

$$x_R(\sigma) = x(\pi-\sigma) = r_0 + \sqrt{2}\sum_{m=1}^{\infty} r_{2m}\cos(2m\sigma), \qquad 0 \le \sigma \le \frac{\pi}{2}\,, \tag{6.3}$$

and

$$l_{2m} = x_{2m} + \sum_{n=1}^{\infty}\mathcal{T}_{2m,2n-1}x_{2n-1}\,, \qquad r_{2m} = x_{2m} - \sum_{n=1}^{\infty}\mathcal{T}_{2m,2n-1}x_{2n-1}\,, \tag{6.4}$$

with

$$\mathcal{T}_{2n,2m-1} = \frac{4}{\pi}\int_0^{\pi/2} d\sigma \cos 2n\sigma \cos(2m-1)\sigma\ . \tag{6.5}$$

We have seen in the last section that the open string star product in the operator form can be written (up to an infinite constant) as the canonical Moyal product in terms of coordinates (x_κ, y_κ) with a continuous label $\kappa > 0$. Here we make a direct connection with (6.1).

By inverting (4.11), (4.12) and using (5.1) we can go back to the standard Fock space basis and interpret the open string star product as the Moyal product in terms of coordinates $(\{x_{2n}\}, \{p_{2m-1}\})$

$$[x_{2n}, p_{2m-1}]_* = i\Theta^{2n,2m-1}\,, \qquad n, m \ge 1\,, \tag{6.6}$$

where from equations (5.1), (4.11), (4.12), the noncommutative parameter Θ is given by

$$\Theta^{2n,2m-1} = -2\sqrt{\frac{2m-1}{2n}}\int_{-\infty}^{+\infty} v_{2m-1}(\kappa)v_{2n}(\kappa)\tanh\left(\frac{\pi\kappa}{4}\right)d\kappa\,. \tag{6.7}$$

Note that here p_{2m-1} play the role of "coordinates" while x_{2m-1} are interpreted as the corresponding "momenta". Using (6.6) and recalling that x_{2m-1} are the momenta conjugate to the noncommutative coordinates p_{2m-1}, we see that (6.1)–(6.4) precisely have the form of the Moyal product in the mixed coordinate and momentum basis (2.11)–(2.13) if we identify [6]

$$\frac{1}{2}\Theta^{2n,2m-1} = \mathcal{T}_{2n,2m-1} \qquad n, m \ge 1\,. \tag{6.8}$$

[6] In this paper we ignore the zero mode part, i.e. $n = 0$ elements of the matrix.

We will now prove (6.8) by considering generating functions. We define

$$\begin{aligned}\tilde{\mathcal{B}}(z,w) :&= \frac{1}{2}\sum_{n,m\geq 1} z^{2n} w^{2m-1}\Theta^{2n,2m-1}\\ &= -\frac{1}{2}\frac{w}{1+w^2}\int_{-\infty}^{+\infty} d\kappa \frac{\tanh(\pi\kappa/4)}{\sinh(\pi\kappa/2)}\cosh\kappa v(1-\cosh\kappa u)\,,\end{aligned} \tag{6.9}$$

with $u = \tan^{-1} z$ and $v = \tan^{-1} w$. In the second line above we have inserted the generating function (3.4) for v_n and the explicit expression for $\theta(\kappa)$. Similarly we consider

$$\begin{aligned}\mathcal{B}(z,w) :&= \sum_{n,m\geq 1}^{\infty} z^{2n} w^{2m-1}\mathcal{T}_{2n,2m-1}\\ &= \frac{z^2 w}{4\pi i}\oint \frac{d\xi}{\xi^{1/2}}\left(\frac{\xi}{1-z^2\xi}+\frac{1}{\xi-z^2}\right)\left(\frac{1}{1-w^2\xi}+\frac{1}{\xi-w^2}\right),\end{aligned} \tag{6.10}$$

where the integral extends from $\xi = e^{-i\pi}$ along the unit circle to $\xi = e^{+i\pi}$. It can be shown by evaluating the integrals (6.9) and (6.10) that indeed

$$\tilde{B}(z,w)=B(z,w)=\frac{i}{\pi}\frac{(1-w^2)z}{(z^2-w^2)(1-z^2w^2)}\left[z(1+w^2)\log\left(\frac{1+iw}{1-iw}\right)-w(1+z^2)\log\left(\frac{1+iz}{1-iz}\right)\right] \tag{6.11}$$

The first integral (6.9) is easily done by closing the contour in the upper half plane. To do the second integral we deform the contour, picking up the residues at $\xi = z^2, w^2$ to an integral around the cut from 0 to -1. To do the integral around the cut one needs:

$$\int_0^1 \frac{dx}{x^{1/2}}\frac{1}{1+a^2x} = \frac{1}{ia}\log\left(\frac{1+ia}{1-ia}\right). \tag{6.12}$$

This completes the verification of (6.8).

Finally let us compare the noncommutativity parameter (6.7) with that given by Bars in [4, eq. (33)]:

$$\left[x_{2m}^{(\mathrm{Bars})\ \mu}, p_{2n}^{(\mathrm{Bars})\ \nu}\right] = i\eta^{\mu\nu}\delta_{m,n}\,. \tag{6.13}$$

To compare with our results we note that $x_{2n}^{\mathrm{here}} = x_{2n}^{(\mathrm{Bars})}$ and from [4], eq. (10):

$$p_{2m-1}^{\mathrm{here}} = 2\sum_{n=1}^{\infty} p_{2n}^{(\mathrm{Bars})}\mathcal{T}_{2n,2m-1}\,, \tag{6.14}$$

and therefore equation (6.8) indeed shows they are equivalent.

While this is a satisfying result, it is also somewhat puzzling. We have found a continous spectrum of noncommutativity parameters, including $\theta(\kappa = 0) = 0$, in contrast to Bars' result (6.13). Since, as we have just shown, $\mathcal{T}$ is essentially the same as Θ, and since the latter matrix has a zeromode, it follows that $\mathcal{T}$ has a zeromode, and hence we cannot invert the relation (6.14).

It might sound ridiculous to say that $\mathcal{T}$ is not invertible, since one can introduce the explicit matrix $\mathcal{R}$ given by

$$\mathcal{R}_{2m-1,2n} = \mathcal{T}_{2n,2m-1} - (-1)^n \mathcal{T}_{0,2m-1}\,, \tag{6.15}$$

which satisfies[7]

$$(\mathcal{R}\mathcal{T})_{2m-1,2k-1} = \delta_{m,k}\,, \qquad (\mathcal{T}\mathcal{R})_{2n,2k} = \delta_{n,k}\,. \tag{6.16}$$

The sums implicit in (6.16) are absolutely convergent; doesn't this mean that $\mathcal{T}$ is invertible? Not necessarily, because the spectrum of an operator depends on the linear space on which it is defined. For example, while $\mathcal{T}$ is invertible on the Hilbert space ℓ_2 of square-integrable sequences it is not invertible on the Banach space ℓ_∞ of bounded sequences. (The eigenmodes $v_n(\kappa)$ are elements of this latter space.) Which linear space one should use in formulating string field theory is a matter of physical definition. Since the zero mode of $\mathcal{T}$ appears to be physically meaningful, and is moreover important in the $\alpha' \to 0$ limit [12] we believe we should take seriously the zero mode of $\mathcal{T}$, and not make redefinitions which obscure it. Further evidence for this point of view will appear in the next section.[8]

7. String field theory as NC field theory

The results in previous sections allow us to rewrite the cubic interaction as

$$\int \Psi * \Psi * \Psi\,, \tag{7.1}$$

where $*$ is now the continuous tensor product of Moyal products, and $\int$ is the standard integration in terms of the Moyal coordinates.[9] We would now like to write the entire matter part of the bosonic string field theory action,

$$S = \int \frac{1}{2}\Psi(L_0 - 1)\Psi + \frac{1}{3}\Psi * \Psi * \Psi\,, \tag{7.2}$$

as a noncommutative field theory action. Hence we now turn to a detailed discussion of the kinetic term.

We are going to find difficulties in writing a well-defined kinetic term when we use variables in which the $*$-product is simple, therefore we begin with a general discussion. The string field wave function Ψ can be written as a function of infinitely many variables in many ways. The standard way is to use the variables x_n in the Fourier expansion of $X(\sigma)$. However, we have seen that one may wish to Fourier transform some variables and take nontrivial linear combinations of the resulting coordinates. Therefore, let us simply

[7] Essentially equivalent identities occur in the paper of Gross and Taylor [8] concerning the construction of orthogonal transformations from their X_{eo} and X_{oe}.

[8] We should mention that Bars and Matsuo [18] have also addressed subtleties associated with this zero mode, and have suggested that these be thought of in terms of explicit "associativity anomalies".

[9] We expect that inclusion of the ghosts into this discussion should be possible.

assume that the string field is a function of some infinite collection of variables z^i such that the kinetic operator can be written as

$$L_0 = \sum_{ij} g^{ij} a_i^\dagger a_j \, . \tag{7.3}$$

(i, j here can be continuous, as with the κ basis, or discrete). We assume the oscillators a_i act as:

$$\begin{aligned} a_i \Psi &= f_i * \Psi + \Psi * g_i \, , \\ a_i^\dagger \Psi &= f_i^* * \Psi + \Psi * g_i^* \, , \end{aligned} \tag{7.4}$$

for some elements f_i, g_i of the algebra. It is then straightforward to check that the kinetic term can be written as:

$$\int \Psi * L_0 \Psi = \int g^{ij} \mathrm{Re}[f_i, \Psi][g_j^*, \Psi] + \int \Psi * \Psi * V \, , \tag{7.5}$$

with

$$V = \frac{1}{2} g^{ij} \left(\{ f_i^* + g_i^*, f_j + g_j \}_* + [f_i^*, f_j]_* + [g_i, g_j^*]_* \right) , \tag{7.6}$$

where all products are star products.

We are now going to examine three choices of basis in which the star product is reasonably simple. We will find difficulties with all three bases. Indeed, the generality of the difficulties can be seen by looking at invariants under linear changes of coordinate, such as the spectrum of linear operators.

For our first choice of coordinates, we take

$$z^\mu = \left(x_{2n} \quad p_{2m-1} \right) \, . \tag{7.7}$$

Here μ is an index running over positive even, then positive odd integers. In terms of these coordinates the standard string field kinetic term is given by

$$\begin{aligned} L_0 &= \sum_{n \geq 1} n a_n^\dagger a_n \\ &= \frac{1}{2} \sum \left(p_n^2 + n^2 x_n^2 \right) \\ &= \frac{1}{2} \sum \left(- \frac{\partial^2}{\partial x_{2n}^2} + (2n)^2 x_{2n}^2 \right) + \frac{1}{2} \sum \left(-(2m-1)^2 \frac{\partial^2}{\partial p_{2m-1}^2} + p_{2m-1}^2 \right) \\ &= \frac{1}{2} \sum_\mu \left[- g_{(1)}^{\mu\nu} \partial_\mu \partial_\nu + (g_{(2)})_{\mu\nu} z^\mu z^\nu \right] , \end{aligned} \tag{7.8}$$

where we have introduced the metrics:

$$g_{(1)}^{\mu\nu} = \delta_{2n,2n'} \oplus (2m-1)^2 \delta_{2m-1,2m'-1} \, , \tag{7.9}$$

$$\left(g_{(2)} \right)_{\mu\nu} = (2n)^2 \delta_{2n,2n'} \oplus \delta_{2m-1,2m'-1} \, . \tag{7.10}$$

Quite generally, given an invertible noncommutativity parameter

$$[z^\mu, z^\nu] = i\Theta^{\mu\nu}\,, \tag{7.11}$$

one can rewrite a derivative as a star commutator,

$$\partial_\mu \Psi = -i\Theta^{-1}_{\mu\nu}[z^\nu, \Psi]_*\,. \tag{7.12}$$

Since the string field theory trace $\int$ is the same as the noncommutative field theory trace $\int$, we can use cyclicity of the trace, (and the fact that the metrics and Θ are constant, as functions of z^μ) to obtain

$$\int \Psi L_0 \Psi = \frac{1}{2}\int [z^\mu, \Psi]_* Q_{\mu\nu}[z^\nu, \Psi]_* + (g_{(2)})_{\mu\nu}(z^\mu z^\nu) * (\Psi^2)\,, \tag{7.13}$$

where

$$Q_{\mu\nu} = \Theta^{-1}_{\mu\rho} g^{\rho\lambda}_{(1)} \Theta^{-1}_{\lambda\nu} = \sum_n \tilde{\Theta}^{-1}_{\mu,2n}\tilde{\Theta}^{-1}_{2n,\nu} + \sum_m \tilde{\Theta}^{-1}_{\mu,2m-1}(2m-1)^2\tilde{\Theta}^{-1}_{2m-1,\nu}\,, \tag{7.14}$$

and all products are commutative products unless explicitly indicated as $*$-products.

This presentation of the action is very close to that for a generic noncommutative field theory [7], with the slight difference that the potential explicitly breaks translation symmetry. One might therefore hope to adapt results such as the existence of solitonic solutions very directly to string field theory, thus justifying the identification of D-branes with string field theory solitons.

The form (7.13) is actually somewhat deceptive for two reasons. First, as we have noted, Θ has a zero mode, so it does not have an inverse. Of course this is a standard situation in noncommutative field theory, and one can easily enough deal with it by not making the substitution (7.12) for commuting coordinates. What is potentially more problematic here is that Θ has a continuous spectrum around zero. This opens the possibilities that one will not be able to separate out the zero mode usefully, or that the rewriting (7.12) might lead to divergences even for the noncommuting coordinates with arbitrarily small θ.

A more fundamental difficulty, however, is that the products of the operators Θ, $g_{(1)}$ and $g_{(2)}$ which arise in evaluating the kinetic term derived from (7.13), are ill defined. Consider for example the potential term; evaluating it on a generic string functional will involve acting with operators such as

$$\Theta g_{(2)} \Theta g_{(2)}\,. \tag{7.15}$$

Unfortunately, this operator does not really exist – and this is a basis-independent statement.

The problem becomes clear when one attempts to express L_0 in the κ-basis as an integral kernel. That is, we search for a function $K(\kappa, \kappa')$ such that

$$[L_0, a_\kappa] = \int_{-\infty}^{+\infty} d\kappa' K(\kappa, \kappa') a_{\kappa'}\,. \tag{7.16}$$

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Unfortunately the formal solution to (7.16),

$$K(\kappa,\kappa') = \sum n v_n(\kappa) v_n(\kappa') \tag{7.17}$$

does not exist because the series does not converge. This is apparent from the large n, fixed κ, asymptotics of $v_n(\kappa)$ (N below is given by (3.5)):

$$\begin{aligned}\sqrt{n+1}\,v_{n+1}(\kappa) &\sim \frac{(-1)^{n/2}}{N(\kappa)^{1/2}}\mathrm{Re}\left[\frac{(2n)^{i\kappa/2}}{\Gamma(1+i\frac{1}{2}\kappa)}\right] \qquad n \text{ even}\,,\\ &\sim \frac{(-1)^{(n+1)/2}}{N(\kappa)^{1/2}}\mathrm{Im}\left[\frac{(2n)^{i\kappa/2}}{\Gamma(1+i\frac{1}{2}\kappa)}\right] \qquad n \text{ odd}\,,\end{aligned} \tag{7.18}$$

and hence the series (7.17) does not converge. One can check that direct evaluation of (7.15) in the first (n) basis leads to the expressions containing the same nonconvergent series (7.17).

One might imagine that a kernel satisfying (7.16) still exists, but that it cannot be obtained by doing the sum (7.17). This hypothesis can be tested by considering the regularized operator $L_0^q := \sum q^n a_n^\dagger a_n$, where $0 < q < 1$. Then one can indeed find a well-defined kernel satisfying:

$$[L_0^q, a_\kappa] = \int d\kappa' K_q(\kappa,\kappa') a_{\kappa'}\,, \tag{7.19}$$

where L_0^q is a regularized version of L_0. However, $K_q(\kappa,\kappa')$ does not have a well-defined $q \to 1$ limit, as is already shown by the identity

$$K_q(\kappa,0) = \frac{1}{\sqrt{4\pi N}}\frac{q}{1-q^2}\left[\left(\frac{1-q}{1+q}\right)^{i\kappa/2} + \left(\frac{1+q}{1-q}\right)^{i\kappa/2}\right]. \tag{7.20}$$

A similar, but more elaborate, formula exists for the full kernel $K_q(\kappa,\kappa')$, and shows that there is no well-defined $q \to 1$ limit for all values of κ,κ'.

Having found that our bases are problematic, let us turn to the basis advocated in [4]. Rather than using x_{2m}, p_{2m-1} we now use $\bar{y}, y_{2m}$ and p_{2m}^{Bars}, where $\bar{y} = x_0 + \sqrt{2}\sum(-1)^n x_{2n}$ and $y_{2n} = x_{2n}$. Since

$$\frac{d}{dx_0} = \frac{d}{d\bar{y}}\,, \tag{7.21}$$

$$\frac{d}{dx_{2n}} = \frac{d}{dy_{2n}} + \sqrt{2}(-1)^n\frac{d}{d\bar{y}}\,, \tag{7.22}$$

the zeromomentum sector with respect to x_0 coincides with that with respect to $\bar{y}$. We restrict ourselves to this sector since nonzero momentum with respect to $\bar{y}$, with finite momentum with respect to y_{2m} appears to be problematic.

Now, using the identities

$$\sum \mathcal{T}_{2n,2m-1}\mathcal{T}_{2n',2m-1} = \delta_{n,n'}\,, \tag{7.23}$$

valid for $n, n' > 0$, and

$$\sum_{m\geq 1}(2m-1)^2\mathcal{R}_{2m-1,2n}\mathcal{R}_{2m-1,2n'} = (2n)^2\delta_{n,n'}\,, \tag{7.24}$$

we convert L_0 to

$$L_0 = \frac{1}{2}\sum\left(-\frac{\partial^2}{\partial x_{2n}^2} + (2n)^2 x_{2n}^2\right) + \frac{1}{2}\sum\left(-m^2\frac{\partial^2}{\partial\left(p_{2m}^{\mathrm{Bars}}\right)^2} + \left(p_{2m}^{\mathrm{Bars}}\right)^2\right). \tag{7.25}$$

Evidently, we have not reproduced the standard perturbative string spectrum! Instead of a single tower of oscillators with one oscillator for each integral frequency, we have one oscillator for every odd frequency and two oscillators for every even frequency. What has happened?

Once again, one can trace the difficulty back to the fact that the spectrum of an operator is sensitive to the linear space on which it is defined. As noted in [12], identities such as (7.24) imply, naively, that one can change the set of eigenvalues of a diagonal matrix by an invertible transformation. This absurdity results from neglecting subtleties in dealing with unbounded operators (e.g. see [17]). Evidently, such niceties of functional analysis are relevant for string field theory.

8. Discussion and new directions

We can summarize the main conclusions of our discussion as follows.

- We have a Moyal description of the star product which arises unambiguously from the standard mode basis by a unitary transformation, equation (3.15), implemented by a well-behaved infinite dimensional matrix.
- It leads to a tensor product of algebras with a continuous spectrum of θ's including zero. While a finite number of commutative modes can be incorporated into an NC field theory framework by treating them specially, a continuous spectrum around zero is something new and poses interesting questions.
- The theta spectrum is bounded above. We believe this is physically meaningful, since we did a well-defined unitary change of basis. It means that the open string star noncommutativity has a definite "maximal range" in the usual NC field theory sense, where space nonlocality is related to the momentum as $\Delta x = \theta p$. This seems most intriguing to us and its physical implications deserve more thought.

We should remind the reader that the inner product with respect to which our transformation is unitary is equation (5.5); this is the quadratic form which (for example) appears in the ground state (or any Fock state) wave functional and thus can be said to govern the α' fluctuations of a free string. The boundedness of $\theta(\kappa)$ is therefore the statement that the noncommutativity of the interaction term is comparable to (or less than, for modes with small κ) the quantum uncertainty of the free string.

We also note that the precise relation between the Witten and Moyal products (4.4) involved an overall multiplicative factor, (5.24). Of course, we computed this only for the matter sector and the ghosts will clearly modify this result. Since ultimately we are interested in wavefunctionals for which the open string star product is finite and thus define the open string algebra, this factor must be taken seriously in passing to the Moyal product formulation.

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An important extension of the present work would be to discuss the reformulation of the star product in terms of the Moyal product for arbitrary string functionals, that is, relaxing the zero momentum condition imposed in the present paper. The spectrum of the requisite Neumann matrices is not completely known, but it is expected to include a continuous set of eigenvalues from $[-1/3, 0)$ with the $(-1/3)$ eigenvalue corresponding to the C-even eigenvector discussed in [12, 19]. It is clear that associated to this eigenvector, the corresponding commuting coordinate is the string midpoint. Presumably there is no C-odd eigenvector for this eigenvalue, so the results might take a form rather similar to those in this paper. On the other hand it is known from numerical experiments [19] that the spectrum also includes a pair of degenerate eigenvectors with eigenvalue in the interval $(0, 1)$. The interpretation of the associated Moyal coordinates could be of interest.

One of the more interesting problems left open by the above discussion is the apparent conflict between finding a simple description of the kinetic term in the string field action and the cubic interaction term. In the standard presentation, the kinetic term is simple, but the star product is complicated. In the present paper we have given another presentation of the string field theory action which makes the star product simple, but in which it is difficult to write a well-defined kinetic term. This conflict seems to persist among various choices of coordinates for the string field, though we have by no means ruled out the existence of coordinates which fix this problem, perhaps obtained by more complicated operations such as additional Fourier transforms. It seems important to understand this difficulty better. Since there is no difficulty in defining the regularized operator L_0^q, computations can be done in this presentation, by taking the $q \to 1$ limit only at the end of the computation. Evidently the ill-defined quantities discussed in section 7 cancel out of physical results.

There is some evidence both for and against the idea that the star product on physical states really is singular, in the sense of not defining an algebra of bounded operators. In [12], it was argued that the squeezed state defining the star product in the matter sector involves a quadratic form which is not Hilbert-Schmidt and therefore the open string star product does not give states of finite norm. Here we confirmed this in Moyal coordinates, in equation (5.17). On the other hand, it must be admitted that many sensible computations have been done using level truncation and the unregularized Witten product. It may also be premature to discuss these issues before taking proper account of ghosts and the BRST operator. If it were to turn out that the star product had to be regulated to get a sensible formalism, one would lose the associativity of the algebra (even though the $\kappa = 0$ mode was being treated properly – so this is different from the nonassociativity of [18]). In place of associativity, one would have A_∞ structure [20], which also has a well developed theory, though not nearly so well developed as conventional operator algebras. Such algebras have featured in constructions of open string field theory [21, 22].

The presentation of the star algebra in this paper is an important step in formulating rigorously what should be meant by the K-theory of the string field theory algebra. It is widely expected that D-branes should be topologically classified, in the context of string field theory, in terms of the K-theory of some noncommutative algebra related to a vertex operator algebra. Certainly the K-homology of C^* algebras does appear to be physically relevant to the topological classification of D-branes in the long-distance limit of the theory.

Nevertheless, the precise definition of the K-theory of the entire string field theory algebra has not yet been given. Writing the algebra as a continuous product of standard Heisenberg algebras should open the way to giving such a precise definition. Nevertheless, there is much nontrivial work left to do. The resolution of the difficulties we have pointed out with the kinetic term will have an impact on how this definition is carried out. Moreover, assigning rigorous meaning to the continuous tensor product of C^* algebras might well be nontrivial.

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